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ON THE SPECTRUM OF THE SUM OF GENERATORS FOR A FINITELY GENERATED GROUP

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ABSTRACT

Let Γ be a finitely generated group. In the group algebra $\mathbb{C}[\Gamma]$, form the average h of a finite set S of generators of Γ. Given a unitary representation π of Γ, we relate spectral properties of the operator $\pi(h)$ to properties of Γ and π.

For the universal representation π_{un} of Γ, we prove in particular the following results. First, the spectrum $\mathrm{Sp}(\pi_{un}(h))$ contains the complex number z of modulus one iff $\mathrm{Sp}(\pi_{un}(h))$ is invariant under multiplication by z, iff there exists a character $\chi\colon \Gamma \to \mathbb{T}$ such that $\chi(S) = \{z\}$. Second, for $S^{-1} = S$, the group Γ has Kazhdan's property (T) if and only if 1 is isolated in $\mathrm{Sp}(\pi_{un}(h))$; in this case, the distance between 1 and other points of the spectrum gives a lower bound on the Kazhdan constants. Numerous examples illustrate the results.

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Introduction

Let Γ be a finitely generated group, and let S be a finite generating set. Assume for the moment that $1 \notin S$ and that S is symmetric ($S^{-1} = S$). On the Cayley graph $\mathcal{G}(\Gamma, S)$, consider the nearest neighbour isotropic random walk: the transition probability $M_{\gamma,\gamma'}$ between two vertices γ, γ' of the graph is $|S|^{-1}$ if they are nearest neighbours (namely here if $\gamma\gamma'^{-1} \in S$) and zero otherwise. The matrix $M = (M_{\gamma,\gamma'})_{\gamma,\gamma' \in \Gamma}$ acts naturally as a bounded linear operator on the Hilbert space $l^2(\Gamma)$. In two pioneering papers [Ke1], [Ke2], Kesten started to investigate the relations between properties of the group Γ and spectral properties of transition operators such as M. He showed for example that

$$\frac{2\sqrt{|S|-1}}{|S|} \leq \|M\| \leq 1$$

with equality on the right if and only if Γ is amenable (a condition independent of the choice of S), and with equality on the left if and only if Γ is a free group on a set S_+ such that $S = S_+ \cup (S_+)^{-1}$.

Our purpose here is to modify Kesten's starting point in two ways:

(1) We consider a finite generating set S which need not be symmetric.

(2) We consider an arbitrary unitary representation π of Γ on a Hilbert space $\mathcal{H}_\pi$ rather than only the left regular representation λ of Γ on $l^2(\Gamma)$.

Denote by $\mathbb{C}[\Gamma]$ the complex group algebra. To any representation π of Γ as above is associated a $*$-representation of $\mathbb{C}[\Gamma]$ on the same Hilbert space $\mathcal{H}_\pi$, also denoted by π, and defined by $\pi(f) = \sum_{\gamma\in\Gamma} f(\gamma)\pi(\gamma)$ for all $f \in \mathbb{C}[\Gamma]$.

Given S as in (1), we set

$$h = \frac{1}{|S|}\sum_{s\in S} s \in \mathbb{C}[\Gamma].$$

In case S is symmetric and $1 \notin S$, observe that $\lambda(h)$ is precisely the operator M appearing in Kesten's papers. In the setting of (1) and (2) above, our programme is to investigate the relations between properties of Γ and π on one hand, and spectral properties of $\pi(h)$ on the other hand.

There are several interesting examples of such π, besides the left regular representation λ. One is the **universal representation** π_{un}, which is the direct sum of all cyclic representations of Γ (up to unitary equivalence). Other are characters $\Gamma \to \{z \in \mathbb{C}\colon |z| = 1\}$, including the **trivial representation** χ_1 defined by $\chi_1(\gamma) = 1$ for all $\gamma \in \Gamma$.

To state our first result, we need the following terminology. Given a representation π of Γ on a Hilbert space $\mathcal{H}$ and a finite generating set S as above, we define the Kazhdan constant

$$\kappa(\pi, S) = \inf_{\xi \in \mathbb{S}^1(\mathcal{H})} \max_{s \in S} \|\pi(s)\xi - \xi\|$$

where we denote by $\mathbb{S}^1(\mathcal{H})$ the **unit sphere** $\{\xi \in \mathcal{H}: \|\xi\| = 1\}$. This constant depends on S, but the fact that it is zero or not does not; indeed $\kappa(\pi, S) = 0$ if and only if χ_1 is weakly contained in π. The latter means that $\kappa(\pi, F) = 0$ for any finite subset F of Γ. We denote by $\pi_1^{\perp}$ the restriction of π to the orthogonal complement in $\mathcal{H}$ of the space of vectors fixed by Γ. In Section A, we shall prove.

PROPOSITION I: *Let π be a representation of Γ on a Hilbert space $\mathcal{H}$.*

(1) $1 \in \mathrm{Sp}\ \pi(h)$ *if and only if* χ_1 *is weakly contained in* π.

(2) 1 *is an eigenvalue of* $\mathrm{Sp}\ \pi(h)$ *if and only if* χ_1 *is contained in* π.

(3) *If* $1 \in \mathrm{Sp}\ \pi(h)$ *and* $1 \notin \mathrm{Sp}\ \pi_1^{\perp}(h)$, *then 1 is an isolated point of both* $\mathrm{Sp}\ \pi(h)$ *and* $\mathrm{Sp}\ \pi(\frac{1}{2}(h + h^*))$.

(4) *If* $\kappa(\pi, S) > 0$, *then* $\mathrm{Sp}\ \pi(h)$ *is disjoint from the open disc*

$$\{z \in \mathbb{C}: |z - 1| < \kappa(\pi, S)^2/(2|S|)\}.$$

Assume moreover that $S^{-1} = S$.

(5) *If* 1 *is isolated in* $\mathrm{Sp}\ \pi(h)$, *then* $1 \notin \mathrm{Sp}\ \pi_1^{\perp}(h)$.

(6) *If* $\mathrm{Sp}\ \pi(h) \subset [-1, 1 - \varepsilon]$ *for some* $\varepsilon > 0$ *then* $\kappa(\pi, S) \geq \sqrt{2\varepsilon}$.

We show by an example the assumption $S^{-1} = S$ cannot be removed from (5). Actually we construct an infinite dimensional irreducible representation π of the free group $\mathbb{F}_2$ on a two generator set $S = \{a, b\}$ such that $\mathrm{Sp}\ \pi(h) = \{1\}$. Viewing $\mathbb{F}_2$ as a normal subgroup of index 12 in $\mathrm{SL}(2, \mathbb{Z})$, inducing and decomposing, we obtain an infinite dimensional irreducible representation of $\mathrm{SL}(2, \mathbb{Z})$ which weakly contains χ_1. This answers, almost fortuitously, a question of Bekka in [Bek]; see the end of Section A.

In Sections B and C, we deal with the **peripheral spectrum** of $\pi(h)$, that is with the intersection of $\mathrm{Sp}\ \pi(h)$ with the unit circle $\mathbb{T}$. We prove, for $z \in \mathbb{T}$, results analogous to those of Proposition I for $+1$. We also study rotational symmetries of spectra. Writing $\mathrm{Sp}\, h$ rather then $\mathrm{Sp}\, \pi_{un}(h)$, we give the following characterization (see Propositions 3 and 5).

PROPOSITION II: *Let Γ, S and h be as above and let $\mathcal{G}$ be the Cayley graph of Γ with respect to $S \cup S^{-1}$. The peripheral spectrum of h is a closed subgroup of $\mathbb{T}$. Moreover, for $|z| = 1$, the following are equivalent:*

(i) $z \in \mathrm{Sp}\, h$,

(ii) $\mathrm{Sp}\, h$ *is invariant under multiplication by* z,

(iii) *there is a character* $\chi\colon \Gamma \to \mathbb{T}$ *such that* $\chi(S) = \{z\}$.

Under conditions (i) to (iii), it is also true that $\mathrm{Sp}\,\lambda(h)$ *is invariant under multiplication by* z. *If* $z = -1$, *conditions (i) to (iii) are also equivalent to*

(iv) *the graph* $\mathcal{G}$ *is bicolorable* .

Section D is about **Kazhdan's property (T)**. Given a group Γ and a set of generators S we introduce the Kazhdan constants

$$\kappa(\Gamma, S) = \inf \kappa(\pi, S)$$

where the infimum is taken over all representations π of Γ in a separable Hilbert space which have no nonzero fixed vector, and

$$\hat{\kappa}(\Gamma, S) = \inf\{\kappa(\pi, S)\colon \pi \in \hat{\Gamma}, \pi \neq \chi_1\}$$

where $\hat{\Gamma}$ denotes the **unitary dual** of Γ, namely the set of all (equivalence classes of) irreducible representations of Γ. Here again, the fact that any of these constants are zero or not does not depend on S. Indeed $\kappa(\Gamma, S) > 0$ if and only if Γ had Kazhdan's Property (T): see [Kaz],[HaV]. It is also known that $\hat{\kappa}(\Gamma, S) > 0$ if and only if $\kappa(\Gamma, S) > 0$ [DeK: Lemme 1]. More precisely, one has obviously $\hat{\kappa}(\Gamma, S) \geq \kappa(\Gamma, S)$, possibly with strict inequality, and one may show [BaH] that $\kappa(\Gamma, S) \geq (2|S|)^{-\frac{1}{2}} \hat{\kappa}(\Gamma, S)$. The following result improves a previous result of the third author [Val].

PROPOSITION III: *(1) Assume that Γ has Property (T), and set*

$$\varepsilon = \frac{\hat{\kappa}(\Gamma, S)^2}{2|S|}.$$

For any z in the peripheral spectrum of h (in particular for $z = 1$), the set

$$\mathrm{Sp}\, h \cap \{w \in \mathbb{C}\colon 0 < |w - z| < \varepsilon\}$$

is empty. Moreover, the cardinality of the peripheral spectrum is at most equal to $\pi/\mathrm{Arcsin}\,(\varepsilon/2)$.

(2) We assume that $S^{-1} = S$ and we choose a real number $\varepsilon > 0$. Suppose moreover that

either $\mathrm{Sp}\ h \subset [-1, 1-\varepsilon] \cup \{1\}$

or $-1 \in \mathrm{Sp}\ h \subset \{-1\} \cup [-1+\varepsilon, 1]$.

Then Γ has Property (T) and $\kappa(\Gamma, S) \geq \sqrt{2\varepsilon}$.

In particular, when $S^{-1} = S$, we see that Γ has Property (T) if and only if 1 is an isolated point in Sp h.

In the final Section E of this paper, we consider a discrete group Γ with Property (T). It was shown by Wang and more recently reproved by Wassermann that Γ has at most finitely many (inequivalent) irreducible representations of any given dimension $m < \infty$ (see Theorems 2.5 and 2.6 of [Wan] and Corollary 2 of [Was]). We apply the preceding theory to obtain an asymptotic bound for the number $\mathrm{Irrep}_\Gamma(m)$ of irreducible representations of Γ of degree at most m, namely:

PROPOSITION IV: *Let Γ be a discrete Kazhdan group, with a given finite symmetric generating subset S; then*

$$\mathrm{Irrep}_\Gamma(m) = 0(e^{Am^2})$$

for some constant A depending on Γ and S .

A. The spectrum near 1

We begin by briefly discussing group C^*-algebras; although they do not play a fundamental role in this paper, they do provide a convenient framework.

Let π be a representation of the group Γ on a Hilbert space $\mathcal{H}$ (in this paper, all representations are unitary). The norm closure of $\pi(\mathbb{C}[\Gamma])$ in the algebra of all bounded linear operators on $\mathcal{H}$ is a C^*-algebra denoted by $C^*_\pi(\Gamma)$. For example, if π is the universal representation π_{un} of Γ, we obtain the **full C^*-algebra** of Γ. We shall follow common practice and denote it simply by $C^*(\Gamma)$. It has the following universal property: every representation π of Γ extends to a $*$-representation $C^*(\Gamma) \to C^*_\pi(\Gamma)$, again denoted by π [Dix: 13.9.3]. Considering the left regular representation λ, we obtain the **reduced C^*-algebra** $C^*_\lambda(\Gamma)$. It follows from a standard result of Hulanicki [Ped: 7.3.9] that the canonical $*$-homomorphism λ: $C^*(\Gamma) \to C^*_\lambda(\Gamma)$ is an isomorphism if and only if Γ is amenable.

From now on we assume that Γ is given together with a finite set S of generators and we consider the operator

$$h = \frac{1}{|S|}\sum_{s\in S} s \in C^*(\Gamma).$$

Clearly $\|h\| \le 1$, and h is self-adjoint if and only if S is symmetric (i.e. $S = S^{-1}$). We denote by Sp x the **spectrum** of an element x of a C^*-algebra.

PROPOSITION 1: *If $\chi\colon \Gamma \to \{z \in \mathbb{C}\colon |z| = 1\}$ is a character of Γ, then $\chi(h) \in \mathrm{Sp}\ h$. In particular the spectrum of h always contains* 1.

Proof: If $x = \sum_{\gamma\in\Gamma} z_\gamma \gamma \in C^*(\Gamma)$ is an element of $\mathbb{C}[\Gamma]$, then

$$\chi(x) = \sum_{\gamma\in\Gamma} z_\gamma \chi(\gamma).$$

In particular $\chi(h - \chi(h)) = 0$ and therefore $h - \chi(h)$ is not invertible in $C^*(\Gamma)$. The second assertion follows with $\chi = \chi_1$. ∎

Let $\mathcal{H}$ be a Hilbert space. We denote by $\mathbb{S}^1(\mathcal{H})$ the unit sphere $\{\xi \in \mathcal{H}\colon \|\xi\| = 1\}$. Let x be an operator on $\mathcal{H}$. Recall that Sp $x \neq \emptyset$ if and only if $\mathcal{H} \neq \{0\}$. We say that x is a **contraction** if $\|x\| \le 1$. In the following four lemmas, we collect standard material.

LEMMA 1: *Let x be a contraction on a Hilbert space $\mathcal{H}$, let $\xi \in \mathbb{S}^1(\mathcal{H})$ and let ε be a real number, $\varepsilon \ge 0$.*

(1) *If $\|x\xi - \xi\| \le \varepsilon$, then $\|\frac{1}{2}(x + x^*)\xi - \xi\| \le \sqrt{2\varepsilon}$.*
(2) *If $\|\frac{1}{2}(x + x^*)\xi - \xi\| \le \varepsilon$, then $\|x\xi - \xi\| \le \sqrt{2\varepsilon}$.*

Proof: If $\|x\xi - \xi\| \le \varepsilon$, then $\|x\xi\| \ge \|\xi\| - \|x\xi - \xi\| \ge 1 - \varepsilon$ and

$$(1-\varepsilon)^2 + 1 - 2\mathrm{Re}\langle\xi|x\xi\rangle \le \|x\xi - \xi\|^2 \le \varepsilon^2$$

so that $\mathrm{Re}\langle\xi|x\xi\rangle \ge 1 - \varepsilon$ and

$$1 - \langle\xi|\frac{1}{2}(x + x^*)\xi\rangle = 1 - \mathrm{Re}\langle\xi|x\xi\rangle \le \varepsilon.$$

Consequently

$$\begin{aligned}\|\frac{1}{2}(x + x^*)\xi - \xi\|^2 &\le \|x\|^2 + 1 - 2\langle\xi|\frac{1}{2}(x + x^*)\xi\rangle\\ &\le 2(1 - \langle\xi|\frac{1}{2}(x + x^*)\xi\rangle)\\ &\le 2\varepsilon\end{aligned}$$

and (1) holds.

Similarly, under the hypothesis of (2), one has $\mathrm{Re}\langle\xi|x\xi\rangle \geq 1-\varepsilon$ and

$$\|x\xi-\xi\|^2 \leq 2(1-\mathrm{Re}\langle\xi|x\xi\rangle) \leq 2\varepsilon$$

so that (2) holds. ∎

Remark: It is easy to see (with $\mathcal{H}=\mathbb{C}$) that the constant $\sqrt{2\varepsilon}$ in Claim (2) is best possible. However the constant in Claim (1) is not. We do not know the best constant here.

LEMMA 2: *Let x be a contraction on a Hilbert space $\mathcal{H}$.*

(1) $\mathrm{Ker}(x-1)=\mathrm{Ker}(\frac{1}{2}(x+x^*)-1)=\mathrm{Ker}(x^*-1)$.

(2) $1\in \mathrm{Sp}\, x$ *if and only if there exists a sequence* (ξ_j) *of vectors in* $\mathbb{S}^1(\mathcal{H})$ *such that* $\|x\xi_j-\xi_j\|\to 0$ *as* $j\to\infty$.

(3) $1\in \mathrm{Sp}\, x \Leftrightarrow 1\in \mathrm{Sp}(\frac{1}{2}(x+x^*)) \Leftrightarrow 1\in \mathrm{Sp}\, x^*$.

Proof: Claim (1) follows from Lemma 1 with $\varepsilon=0$.

To prove the non-trivial implication in Claim (2), suppose that $1\in\mathrm{Sp}\,x$. If the range of $x-1$ is not dense in $\mathcal{H}$, then $\mathrm{Ker}(x^*-1)\neq\{0\}$. But $\mathrm{Ker}(x-1)=\mathrm{Ker}(x^*-1)$ by (1), so that any sequence in $\mathrm{Ker}(x-1)\cap\mathbb{S}^1(\mathcal{H})$ does the job. If the range of $x-1$ is dense in $\mathcal{H}$, then $\inf\{\|(x-1)\xi\|\colon \xi\in\mathbb{S}^1(\mathcal{H})\}=0$ (otherwise $x-1$ would be invertible), and the existence of an appropriate sequence (ξ_j) is again obvious.

Claim (3) is a straightforward consequence of Claim (2) and of Lemma 1. ∎

LEMMA 3: *Consider a Hilbert space $\mathcal{H}$, a vector $\xi\in\mathbb{S}^1(\mathcal{H})$, a real number $\varepsilon\geq 0$, an integer $n\geq 1$, a sequence $y_1,\ldots,y_n$ of contractions on $\mathcal{H}$, and set*

$$x=\frac{1}{n}(y_1+\cdots+y_n).$$

If $\|x\xi-\xi\|\leq\varepsilon$, then $\|y_j\xi-\xi\|\leq\sqrt{2n\varepsilon}$ for all $j\in\{1,\ldots,n\}$. In particular

$$\mathrm{Ker}(x-1)=\bigcap_{1\leq j\leq n}\mathrm{Ker}(y_j-1).$$

Proof: As in the proof of Lemma 1, we compute

$$1-\varepsilon\leq \mathrm{Re}\langle\xi|x\xi\rangle=\frac{1}{n}\sum_{k=1}^{n}\mathrm{Re}\langle\xi|y_k\xi\rangle.$$

As $\mathrm{Re}\langle\xi|y_k\xi\rangle \leq 1$ for all $k \in \{1,\dots,n\}$, it follows that

$$\mathrm{Re}\langle\xi|y_j\xi\rangle \geq 1 - n\varepsilon$$

and thus

$$\|y_j\xi - \xi\|^2 \leq 2n\varepsilon$$

for all $j \in \{1,\dots,n\}$. ∎

LEMMA 4: *Given Γ, S and π as above, $\kappa(\pi, S) = 0$ if and only if the trivial representation χ_1 of Γ is weakly contained in π.*

Proof: For each integer $n \geq 0$, denote by B_n the set of those $\gamma \in \Gamma$ for which there exists a sequence $s_1,\dots,s_n$ of generators in $S \cup S^{-1}$ such that $\gamma = s_1 \cdots s_n$.

Consider a real number $\varepsilon > 0$ and a vector $\xi \in \mathbb{S}^1(\mathcal{H})$ such that

$$\max_{s\in S} \|\pi(s)\xi - \xi\| \leq \kappa(\pi, S) + \varepsilon.$$

For any $\gamma = s_1 \cdots s_n \in B_n$, one has

$$\begin{aligned}\|\pi(\gamma)\xi - \xi\| &\leq \sum_{j=1}^{n} \|\pi(s_1 \cdots s_{j-1})(\pi(s_j)\xi - \xi)\| \\ &\leq n(\kappa(\pi, S) + \varepsilon).\end{aligned}$$

It follows that

$$\inf_{\xi\in \mathbb{S}^1(\mathcal{H})} max_{\gamma\in B_n} \|\pi(\gamma)\xi - \xi\| \leq n\kappa(\pi, S).$$

In particular, the lemma is nothing but a reformulation of the definitions. ∎

Remark: It follows from Lemma 4 that the condition $\kappa(\pi, S) = 0$ does not depend on the finite set S of generators.

Proof of Proposition I in the Introduction: Claim (1) follows from Lemmas 2(2) and 3 applied to

$$x = \pi(h) = \frac{1}{|S|}\sum_{s\in S} \pi(s),$$

and from Lemma 4.

Claim (2) follows from the last assertion of Lemma 3.

Let us denote by $\pi_1^=$ the restriction of π to the space $\mathcal{H}_1^=$ of $\pi(\Gamma)$-fixed vectors.

Observe that $\mathrm{Sp}\ \pi_1^{=}(h) = \mathrm{Sp}\ \pi_1^{=}(h) \cup \mathrm{Sp}\ \pi_1^{\perp}(h)$. Observe also that $\mathrm{Sp}\ \pi_1^{=}(h)$ is either empty, if $\mathcal{H}_1^{=} = \{0\}$, or reduced to $\{1\}$, if $\mathcal{H}_1^{=} \neq \{0\}$. The first conclusion of Claim (3) follows from this. Moreover $1 \notin \mathrm{Sp}\ \pi_1^{\perp}(h)$ implies that $1 \notin \mathrm{Sp}\ \pi_1^{\perp}(\frac{1}{2}(h+h^*))$ by Lemma 2 (3), and the second conclusion of Claim (3) follows similarly.

Write δ for $\kappa(\pi,S)^2/(2|S|)$ and choose $\xi \in \mathbb{S}^1(\mathcal{H})$. By the definition of $\kappa(\pi,S)$, there exists $s \in S$ such that

$$\|\pi(S)\xi - \xi\| \geq \kappa(\pi,S) = \sqrt{2|S|\delta}.$$

Lemma 3 shows that $\|\pi(h)\xi - \xi\| \geq \delta$. Choose now $w \in \mathbb{C}$ such that $|w| < \delta$. Then

$$\|\pi(h)\xi - (1+w)\xi\| \geq \delta - |w|.$$

As π is a unitary representation, one has $\kappa(\pi, S^{-1}) = \kappa(\pi,S)$. Thus one has also

$$\|\pi(h^*)\xi - (1+\bar{w})\xi\| \geq \delta - |w|.$$

Since these hold for all $\xi \in \mathbb{S}^1(\mathcal{H})$, the operator $\pi(h) - (1+w)$ is invertible, and Claim (4) holds.

If $S^{-1} = S$ and 1 is isolated in $\mathrm{Sp}\ \pi(h)$, we may consider the nonzero spectral projection p of the self-adjoint operator $\pi(h)$ corresponding to the isolated point 1 in the spectrum of $\pi(h)$. The restriction of $\pi(h)$ to $p\mathcal{H}$ coincides with the identity, and the restriction of $\pi(h)$ to $(1-p)\mathcal{H}$ has its spectrum disjoint from $\{1\}$. In other words, one has $p\mathcal{H} = \mathcal{H}_1^{=}$ and $1 \notin \mathrm{Sp}\ \pi_1^{\perp}(h)$.

The hypothesis of (6) and spectral theory imply that, for all $\xi \in \mathbb{S}^1(\mathcal{H})$, one has

$$\mathrm{Re}\langle \xi | \pi(h)\xi\rangle = \langle \xi | \pi(h)\xi\rangle \leq 1 - \varepsilon.$$

Thus there exists $s \in S$ such that

$$\mathrm{Re}\langle \xi | \pi(s)\xi\rangle \leq 1 - \varepsilon$$

and this implies that $\|\pi(s)\xi - \xi\|^2 = 2(1 - \mathrm{Re}\langle \xi|\pi(s)\xi\rangle) \geq 2\varepsilon$. ∎

Remark: In Proposition I (4), assume moreover that $S^{-1} = S$ and that $\pi(S)$ does not contain any element of order 2. Then

$$\mathrm{Sp}\ \pi(h) \subset [-1, 1 - \kappa(\pi,S)^2/|S|].$$

The point is that $2|S|$ can be replaced by $|S|$. For if $s \in S$ is chosen with $\|\pi(s)\xi - \xi\|^2 = 2 - \langle(\pi(s) + \pi(s^{-1}))\xi|\xi\rangle \geq \kappa(\pi, S)^2$ then $\langle(\pi(s) + \pi(s^{-1}))\xi|\xi\rangle \leq 2 - \kappa(\pi, S)^2$, so that $\langle\pi(h)\xi|\xi\rangle \leq 1 - \kappa(\pi, S)^2/|S|$.

SOME CONSEQUENCES OF PROPOSITION I (1).

(a) When π is the left regular representation λ of Γ, Proposition I (1) gives Kesten's characterization of amenability [Ke2]:

$$1 \in \mathrm{Sp}\ \lambda(h) \quad \text{if and only if } \Gamma \text{ is amenable}.$$

Indeed, Γ is amenable if and only if χ_1 is weakly contained in λ, by a classical result of Hulanicki [Hul]. For a generalization of Proposition I (1) see also Theorem 1 in [SoW].

(b) For any representation π of Γ, Bekka has introduced a notion of amenability, and π is amenable if and only if χ_1 is weakly contained in $\pi \otimes \bar{\pi}$, where $\bar{\pi}$ denotes the conjugate of π. (See [Bek: Def. 1.1 and Th. 5.1]; for $\bar{\pi}$, see e.g. [Dix: 13.1.5].) As $\lambda \otimes \bar{\lambda}$ is weakly equivalent to λ [Dix: 13.11.3], it follows that λ is amenable if and only if Γ is amenable. From Proposition I (1) of Proposition 2, one has

$$1 \in \mathrm{Sp}((\pi \otimes \bar{\pi})(h)) \quad \text{if and only if } \pi \text{ is amenable}.$$

For example, let α be the adjoint representation of Γ on $l^2(\Gamma - \{1\})$ defined by

$$(\alpha(\gamma)\xi)(\gamma_1) = \xi(\gamma^{-1}\gamma_1\gamma);$$

then Theorem 2.4 of [Bek] states that $1 \in \mathrm{Sp}((\alpha \otimes \bar{\alpha})(h))$ if and only if Γ is **strongly inner amenable** (namely "inner amenable" in the sense of [BeH]).

Similarly, one can see that 1 is an eigenvalue of $(\pi \otimes \bar{\pi})(h)$ if and only if π has a finite dimensional subrepresentation (see the remark following Lemma 9 below).

Remark on Laplacians: Let us now explain how h is related to a "combinatorial Laplacian". Let first $\mathcal{G}$ be a **graph** without loops or multiple edges (the graph is finite or infinite, and non-oriented). We denote by $\mathcal{G}^0$ the set of vertices of $\mathcal{G}$, by $l^2(\mathcal{G}^0)$ the space of square-summable functions from $\mathcal{G}^0$ to $\mathbb{C}$, and by $(e_v)_{v\in\mathcal{G}^0}$ the canonical orthonormal basis for the latter space. The **adjacency matrix** $A = (A_{v,w})_{v,w\in\mathcal{G}^0}$ of $\mathcal{G}$ is defined by $A_{v,w} = 1$ if $v \neq w$ and there is an edge between v and w, and by $A_{v,w} = 0$ otherwise.

We assume from now on that $\mathcal{G}$ has bounded degree, namely that the number of neighbours $D_v = \sum_{w\in\mathcal{G}^0} A_{v,w}$ of a vertex v is bounded by some $D_{\max}$. The degree operator is the bounded self-adjoint operator D on $l^2(\mathcal{G}^0)$, defind by $De_v = D_v e_v$ for all $v \in \mathcal{G}^0$. As $\mathcal{G}$ has bounded degree, the matrix A defines a bounded self-adjoint operator on $l^2(\mathcal{G}^0)$, denoted again by A. The (combinatorial) **Laplacian** $\Delta_{\mathcal{G}}$ of $\mathcal{G}$ is defined by

$$\Delta_{\mathcal{G}} = D - A.$$

In particular, let Γ be a group generated by a symmetric finite set S such that $1 \notin S$. Let $\mathcal{G} = \mathcal{G}(\Gamma, S)$ be the corresponding **Cayley graph**, defined by $\mathcal{G}^0 = \Gamma$ and $A_{\gamma_1,\gamma_2} = 1$ if and only if $\gamma_1\gamma_2^{-1} \in S$. As $\lambda(h) = \frac{1}{|S|}\sum_{s\in S}\lambda(s)$, one has

$$\Delta_{\mathcal{G}} = |S| - \sum_{s\in S}\lambda(s) = |S|(1-\lambda(h)).$$

We define more generally

$$\Delta = |S|(1-h) \in C^*(\Gamma)$$

so that $\Delta_{\mathcal{G}} = \lambda(\Delta)$.

It is important that Δ is a positive element of $C^*(\Gamma)$. We shall express some results in terms of h and some others with Δ. Upon scaling and translating we could easily express all results for both h and Δ.

Examples:

(1) Consider an integer $d \geq 1$, the free abelian group $\mathbb{Z}^d$, and the standard set $S = \{s_1^{\pm}, \ldots, s_d^{\pm}\}$ of generators having $d-1$ coordinates zero and one coordinate 1 or -1. Then $\mathrm{Sp}\,\lambda(\Delta) = [0, 4d]$, as can be viewed as follows; see also [MoW: 7.B].

For each $\alpha \in \mathbb{R}$, define a character χ_α of $\mathbb{Z}^d$ by

$$\chi_\alpha(s_j^{\pm}) = e^{\pm i\alpha}, \quad j = 1, \ldots, d.$$

It follows from Proposition 1 that

$$\frac{1}{2d}\sum_{s\in S}\chi_\alpha(s) = \cos\alpha \in \mathrm{Sp}\, h.$$

Thus $\mathrm{Sp}\, h = [-1, 1]$ and $\mathrm{Sp}\,\Delta = [0, 4d]$. As $\mathbb{Z}^d$ is amenable, $\lambda(h)$ and h have the same spectrum.

(2) Let Γ be the non-abelian free group on a free set S_+ of d generators, and consider the symmetric set $S = S_+ \cup S_+^{-1}$, of cardinality $N = 2d$. Then

$$\mathrm{Sp}\ \lambda(\Delta) = [N - 2\sqrt{N-1}, N + 2\sqrt{N-1}].$$

Observe that the Cayley graph $\mathcal{G}(\Gamma, S)$ is the homogeneous tree of degree N, so that it is easy to give a meaning to the above formula for any integer N, even or odd; this formula is indeed a result of Kesten [Ke1]. See also [Pat: 4.31].

On the other hand the canonical map sending the generators of S to the corresponding generators of $\mathbb{Z}^d$ in example (1) extends to a $*$homomorphism of $C^*(\Gamma)$ onto $C^*(\mathbb{Z}^d)$. It follows that $\mathrm{Sp}(\Delta) = [0, 4d]$.

(3) If Γ is the non-abelian free group on d generators $s_1, \ldots, s_d$ and $S = \{s_1, \ldots, s_d\}$, so that $S^{-1} \neq S$, then

$$\mathrm{Sp}\ h = \{z \in \mathbb{C}\colon |z| \leq 1\}.$$

For if $z \in \mathbb{C}, |z| \leq 1$, there exists $z_j \in \mathbb{C}, |z_j| = 1$ $(j = 1, \ldots, d)$ such that

$$z = \frac{1}{d}(z_1 + \cdots + z_d).$$

Let χ be the character of Γ defined by $\chi(s_j) = z_j$ for $j = 1, \ldots, d$. Then $z = \chi(h) \in \mathrm{Sp}\ h$ by Proposition 1.

(4) Choose an integer $k \geq 2$, let $\Gamma = \mathbb{Z}/k\mathbb{Z} * \cdots * \mathbb{Z}/k\mathbb{Z}$ (r times), identify each factor $\Gamma_j = \mathbb{Z}/k\mathbb{Z}$ of the free product to a subgroup of Γ, and define

$$S = \bigcup_{1 \leq j \leq r} (\Gamma_j - \{1\}).$$

If $r \geq 2$, then

$$\mathrm{Sp}\ h = [-1/(k-1), 1]$$

by [Mlo: Prop. 3]. Note that this set is independent of r.

Let $\rho = ((k-1)(r-1))^{1/2}$ and note that $|S| = r(k-1)$. By [IoP: Th. 3] or by [KuS: Th. 1], one has

$$\mathrm{Sp}\ \lambda(h) = \left\{ \begin{array}{ll} \left[\dfrac{k-2-2\rho}{|S|}, \dfrac{k-2+2\rho}{|S|}\right], & \text{if } k \leq r, \\ \left\{\dfrac{-1}{k-1}\right\} \cup \left[\dfrac{k-2-2\rho}{|S|}, \dfrac{k-2+2\rho}{|S|}\right], & \text{if } k > r. \end{array} \right\}$$

If Γ is the free product $\Gamma_1 * \cdots * \Gamma_r$ of r finite groups $\Gamma_1, \ldots, \Gamma_r$, each of order k, and if $S = \bigcup_{1 \leq j \leq r}(\Gamma_j - \{1\})$ as above, then Sp $\lambda(h)$ remains the same, as noticed in [CS2: §4.2] (the reason being that the respective Cayley graphs are isomorphic).

Notice in this example the appearance of isolated points in Sp $\lambda(h)$. That these also occur in the case of a free product of two finite groups of distinct order was explicitly worked out in [CS1: Theorem 1].

(5) Let Γ be a finite group and let S be a union of conjugacy classes of Γ such that $S^{-1} = S$. It is easy to write the spectra of the corresponding Δ in terms of the values on $s \in S$ of the irreducible characters of Γ. Several examples are worked out in Chapter 8 of [Lub].

(6) Let Γ be a group generated by a finite symmetric set $S^{-1} = S$, so that $\lambda(\Delta)$ is the combinatorial Laplacian of the Cayley graph $\mathcal{G}(\Gamma, S)$. What we have already observed just after the proof of Proposition I can be reformulated as

$$\Gamma \text{ is non-amenable if and only if Sp } \lambda(\Delta) \subset [\varepsilon, 2|S|] \text{ for some } \varepsilon > 0.$$

The last formulation carries over to other graphs, and one may estimate ε in terms of an appropriate "isoperimetric constant" of the graph; see [Ger: Th. 2] and [BMS].

(7) Consider an integer l, the free group Γ on a set $S_+ = \{s_1, \ldots, s_l\}$ of l generators, the symmetric set $S = S_+ \cup (S_+)^{-1}$, and let

$$h = \frac{1}{2l} \sum_{s \in S} s \in C^*(\Gamma)$$

be as usual. Consider also a set $\{\varphi_1, \ldots, \varphi_l\}$ of isometries of the sphere $\mathbb{S}^2$. There is a corresponding action of Γ on $\mathbb{S}^2$ whereby s_j acts as φ_j, and consequently a representation π of Γ on the Hilbert space

$$L_0^2(\mathbb{S}^2) = \left\{ f \in L^2(\mathbb{S}^2) \colon \int_{\mathbb{S}^2} f(x) d\mu(x) = 0 \right\}$$

where μ denotes the rotation-invariant Lebesgue measure on $\mathbb{S}^2$. It can be shown in this case that

$$\|\pi(h)\| \geq \frac{1}{l} \sqrt{2l - 1}.$$

In case $l = (p+1)/2$ for a prime number $p \equiv 1 (\mathrm{mod}\ 4)$, a deep theorem of Lubotzky, Phillips and Sarnak shows that one may choose the φ_j's such that

$$\|\pi(h)\| = \frac{2\sqrt{p}}{p+1}.$$

We refer to [CdV].

EXAMPLES FOR FREE GROUPS AND FOR $SL_2(\mathbb{Z})$.

Let us first construct an irreducible representation π of the free group $\mathbb{F}_2$ on two generators s_1, s_2 which shows that the hypothesis $S^{-1} = S$ cannot be removed from Claim (5) of Proposition I.

Set $\mathcal{H} = L^2[0,1]$. Let V be the Volterra integral operator defined by

$$(V\xi)(\alpha) = \int_0^\alpha \xi(\beta) d\beta$$

for all $\xi \in L^2[0,1]$ and for all $\alpha \in [0,1]$. Set

$$x = (1+V)^{-1}.$$

It is known that Sp $x = \{1\}$, that x has no eigenvector and that $\|x\| = 1$; see Number 148 and Problem 150 of [Hal]. Let $x = u|x|$ be the polar decomposition of x. As $|x|$ is a contraction, one may define two unitary operators

$$v_1 = |x| + i\sqrt{1-|x|^2} \quad \text{and} \quad v_2 = |x| - i\sqrt{1-|x|^2}.$$

If we set $u_1 = uv_1$ and $u_2 = uv_2$, one has

$$x = \frac{1}{2}(u_1 + u_2).$$

Define the representation π of $\mathbb{F}_2$ on $\mathcal{H}$ by

$$\pi(s_j) = u_j, \quad j = 1, 2$$

so that $x = \pi(h)$. Then 1 is of course isolated in Sp $\pi(h) = \{1\}$, but $\mathcal{H} = \mathcal{H}_0^\perp$ by Proposition I(2).

Let us now show that π is irreducible. We shall use the fact (discussed in Number 151 of [Hal]) that any subspace of $\mathcal{H}$ which is invariant under the Volterra integration operator V is one of

$$\mathcal{H}_\alpha = \{\xi \in L^2[0,1] \colon \xi(\beta) = 0 \text{ for almost all } \beta \in [0,\alpha]\}$$

for some $\alpha \in [0,1]$. Similarly, any subspace of $\mathcal{H}$ invariant under V^* is one of

$$\mathcal{H}^\alpha = \{\xi \in L^2[0,1]: \xi(\beta) = 0 \text{ for almost all } \beta \in [\alpha, 1]\}$$

for some $\alpha \in [0,1]$. It follows that the C^*-algebra $C^*(V,V^*)$ generated by V and V^* acts irreducibly on $\mathcal{H}$. As $(1+V)^{-1} = \frac{1}{2}(\pi(s_1)+\pi(s_2))$, the C^*-algebra $C^*_\pi(\mathbb{F}_2)$ contains $C^*(V,V^*)$, and it follows that the representation π is indeed irreducible.

Let us denote by $K(\mathcal{H})$ the algebra of all compact operators on $\mathcal{H}$. Since $C^*(V,V^*)$ is a subalgebra of $K(\mathcal{H})$ that acts irreducibly on $\mathcal{H}$, we have $C^*(V,V^*) = K(\mathcal{H})$ by [Dix: 4.1.6]. From that it follows that $C^*_\pi(\mathbb{F}_2)$ is the linear span of $K(\mathcal{H})$ and of the identity. Consequently, the only irreducible representations of $\mathbb{F}_2$ which are weakly contained in π are χ_1 and π itself [Dix: 4.1.10]. A fortiori χ_1 is weakly contained in $\pi \otimes \bar{\pi}$, hence π is amenable.

It was pointed out to us by M. Cowling that, already in 1951, Yoshizawa has constructed an infinite dimensional irreducible representation of $\mathbb{F}_2$ which is amenable. But Yoshizawa's representation has properties quite different from π, because the former weakly contains any irreducible representations of $\mathbb{F}_2$ [Yos: §3].

Consider now the group $\Gamma = \mathrm{SL}(2,\mathbb{Z})$, together with the subgroup Γ_0 generated by

$$s_1 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad s_2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

It is a classical fact that Γ_0 is free on $\{s_1, s_2\}$, and that it is a normal subgroup of index 12 in Γ. Let σ be the representation induced by π from Γ_0 to Γ. The following result, which is certainly well-known, shows that σ decomposes into finitely many irreducible representations of Γ. We owe the proof to D. Poguntke.

LEMMA 5: *Let Γ_0 be a normal subgroup of finite index of a group Γ, let π be an infinite dimensional irreducible representation of Γ_0 and let σ be the representation induced by π from Γ_0 to Γ. Then there exists a finite number of infinite dimensional irreducible representations $\sigma_1, \ldots, \sigma_n$ of Γ such that $\sigma = \bigoplus_{1\le j\le n} \sigma_j$.*

Proof: Let p: $E \to \Gamma/\Gamma_0$ be the bundle associated with the principal bundle $\Gamma \to \Gamma/\Gamma_0$ and with π, and let $\mathcal{K}$ be the Hilbert space of sections of p, so that σ acts naturally on $\mathcal{K}$. For each $x \in \Gamma/\Gamma_0$, let $\mathcal{K}_x$ be the subspace of sections with support in $\{x\}$. As Γ_0 is normal in Γ, each $\mathcal{K}_x$ is $\sigma(\Gamma_0)$-invariant and $\mathcal{K} =$

$\bigoplus_{x\in\Gamma/\Gamma_0}\mathcal{K}_x$ is a decomposition of $\mathcal{K}$ into infinite dimensional irreducible $(\sigma|\Gamma_0)$-subspaces. It follows that the commutant $\sigma(\Gamma_0)'$ of $\sigma(\Gamma_0)$ in the algebra of all bounded operators on $\mathcal{K}$ is a subalgebra of the algebra of N-by-N matrices, where $N=[\Gamma\colon\Gamma_0]$; moreover any non-zero projection of $\sigma(\Gamma_0)'$ is infinite dimensional.

A fortiori, the commutant $\sigma(\Gamma)'$ of $\sigma(\Gamma)$ is a finite dimensional algebra. Let $\{p_1,\ldots,p_n\}$ be a set of pairwise commuting minimal projections in $\sigma(\Gamma)'$ such that $\sum_{1\leq j\leq n}p_j=1$. For each $j\in\{1,\ldots,n\}$, the image of p_j is an infinite dimensional subspace of $\mathcal{K}$ which is invariant under and irreducible with respect to $\sigma(\Gamma)$, and one may define σ_j as the corresponding subrepresentation of σ. ∎

Let us consider again $\Gamma=SL(2,\mathbb{Z})\supset\Gamma_0=\langle s_1,s_2\rangle$, the representation π of Γ_0 on $L^2([0,1])$ constructed above and the corresponding $\sigma=Ind_{\Gamma_0}^{\Gamma}(\pi)=\bigoplus_{1\leq j\leq n}\sigma_j$. Since the trivial representation of Γ_0 is weakly contained in π, the trivial representation of Γ is weakly contained in σ, and thus also in σ_j for some $j\in\{1,\ldots,n\}$. A fortiori, the trivial representation of Γ is weakly contained in $\sigma_j\otimes\bar{\sigma_j}$, meaning that σ_j is amenable. We have proved:

PROPOSITION 2: *There exists an infinite dimensional irreducible representation of* SL$(2,\mathbb{Z})$ *which is amenable.*

This answers a question of M. Bekka [Bek: Section 5] that was motivated by the fact that, because of property (T), no such representation exists for SL$(n,\mathbb{Z})$ when $n\geq 3$.

B. The peripheral spectrum

We consider again a finitely generated group Γ, a finite set S of generators of Γ and

$$h=\frac{1}{|S|}\sum_{s\in S}s\in C^*(\Gamma).$$

Let π be a unitary representation of Γ on a Hilbert space $\mathcal{H}$. In this section, we study the peripheral spectrum of $\pi(h)$ by means of the techniques of Section A.

Given any multiplicative character χ of Γ (i.e. a homomorphism χ from Γ to the circle group $\mathbb{T}$), we define

$$\kappa_\chi(\pi,S)=inf_{\xi\in S^1(\mathcal{H})}\max_{s\in S}\|\pi(s)\xi-\chi(s)\xi\|$$

Clearly, if $\chi = \chi_1$, then $\kappa_\chi(\pi, S) = \kappa(\pi, S)$; also $\kappa_\chi(\pi, S) = \kappa(\bar{\chi}\pi, S)$ for any character χ, where $\bar{\chi}$ is the conjugate character and the product $\bar{\chi}\pi$ is the pointwise product. We say that χ **is weakly contained in** π if $\kappa_\chi(\pi, S) = 0$.

LEMMA 6: *Let π be a representation of Γ on a Hilbert space $\mathcal{H}$, and let z be a complex number. One has $z \in \mathrm{Sp}\ \pi(h) \cap \mathbb{T}$ if and only if there exists a character χ of Γ such that $\chi(S) = \{z\}$, and χ is weakly contained in π.*

Proof: The "if" part is clear. For the "only if" part, Lemmas 2 and 3 applied to $\bar{z}\pi(h)$ show that there exists a sequence (ξ_j) of vectors in $\mathbb{S}^1(\mathcal{H})$ such that

$$(*) \qquad \lim_{j\to\infty} \|\pi(s)\xi_j - z\xi_j\| = 0$$

for all $s \in S$. Remark the following: given $\gamma, \gamma' \in \Gamma$ such that there exists $k, k' \in \mathbb{Z}$ with

$$\lim_{j\to\infty} \|\pi(\gamma)\xi_j - z^k\xi_j\| = 0,$$
$$\lim_{j\to\infty} \|\pi(\gamma')\xi_j - z^{k'}\xi_j\| = 0,$$

then

$$\lim_{j\to\infty} \|\pi(\gamma\gamma')\xi_j - z^{k+k'}\xi_j\| = 0.$$

This remark shows that the constant function χ: $S \to \mathbb{T}$ of constant value z extends unambiguously to a character χ of Γ. Finally, it follows from $(*)$ that χ is weakly contained in π. ∎

Lemma 6 has the following consequence for the spectra of h in the universal and regular representations.

LEMMA 7: *For any z in the peripheral spectrum of h, one has $\mathrm{Sp}\ h = z \cdot \mathrm{Sp}\ h$ and $\mathrm{Sp}\,\lambda(h) = z \cdot \mathrm{Sp}\ \lambda(h)$.*

Proof: Let χ be a character of Γ such that $\chi(S) = \{z\}$, as in Lemma 6. Then $\chi\pi_{un}$ is unitarily equivalent to π_{un}, so $(\chi\pi_{un})(h) = z\pi_{un}(h)$ has the same spectrum as $\pi_{un}(h)$. Observe now that the preceding argument is valid for any representation π of Γ such that $\chi\pi$ is unitarily equivalent to π for any character χ of Γ. In particular it holds for $\pi = \lambda$ (see [Dix: 13.11.3]).

Definition: Given an integer $n \geq 2$, we say that the set S of generators of Γ is n-coloring (bicoloring for n=2) if there exists z, a primitive n-th root of 1, and $\chi_{S,z}$, a character of Γ mapping S onto $\{z\}$.

Notice that if $\chi_{S,z}$ exists for one primitive n-th root z, then it exists for all. Below, we shall write $\kappa_z(\pi, S)$ for $\kappa_{\chi_S,z}(\pi, S)$.

PROPOSITION 3: *Let $n \geq 2$ be an integer.*

(1) *The peripheral spectrum of h is a closed subgroup of $\mathbb{T}$. It contains the set of n-th roots of 1 if and only if S in n-coloring. It coincides with $\mathbb{T}$ if and only if there exists a homomorphism $\alpha: \Gamma \to \mathbb{Z}$ such that $\alpha(S) = \{1\}$.*

(2) *If S is n-coloring, then the Cayley graph $\mathcal{G}(\Gamma, S \cup S^{-1})$ is n-colorable. The converse is true if $n = 2$, namely:*

$$S \text{ is bicoloring } \Leftrightarrow -1 \in \text{Sp } h \Leftrightarrow \mathcal{G}(\Gamma, S \cup S^{-1}) \text{ is bicolorable .}$$

(3) *The peripheral spectrum of $\lambda(h)$ is either empty or a closed subgroup of $\mathbb{T}$. Moreover, the following are equivalent:*
   - (a) *Γ is not amenable;*
   - (b) *$1 \notin \text{Sp } \lambda(h)$;*
   - (c) *$\text{Sp } \lambda(h) \cap \mathbb{T} = \emptyset$.*

Proof:

(1) The first assertion follows from Lemma 7, the second from Lemma 6. The "if" part of the third assertion is easy; for the "only if" part, select some irrational real number θ, and set $z = e^{2\pi i\theta}$. By Lemma 6, there exists a character χ of Γ such that $\chi(S) = \{z\}$. Identifying the subgroup generated by z with $\mathbb{Z}$, we see that we can take χ for α.

(2) Recall from [Gra:Chapter VII] that a graph $\mathcal{G}$ is n-colorable if there exists a partition $\mathcal{G}^\circ = \mathcal{G}_1^\circ \amalg \cdots \amalg \mathcal{G}_n^\circ$ of its vertices, such that no edge of $\mathcal{G}$ has both its extremities in the same class. From that definition, both assertions are straightforward.

(3) The first assertion follows from Lemma 7. The second follows from the fact that a group Γ is amenable if and only if its regular representation λ weakly contains *some* finite-dimensional representation [Fel: Theorem 3]. ∎

Definition: Fix an integer $n \geq 2$, and z, a primitive n-th root of 1. Let S be a n-coloring finite set of generators of a group Γ, and let $\chi_{S,z}$ be the corresponding character. Given a representation π of Γ on a Hilbert space $\mathcal{H}$, we set

$$\mathcal{H}_{S,z}^{\equiv} = \{\xi \in \mathcal{H}: \pi(\gamma)\xi = \chi_{S,z}(\gamma)\xi \text{ for all } \gamma \in \Gamma\}$$

and we denote by $\mathcal{H}_{S,z}^{\perp}$ the orthogonal complement of $\mathcal{H}_{S,z}^{=}$ in $\mathcal{H}$. The two corresponding subrepresentations of π are denoted by $\pi_{S,z}^{=}$ and $\pi_{S,z}^{\perp}$.

The following result, analogous to Proposition I, can easily be rephrased for any complex number z of modulus 1. For notational convenience, we chose to stick to roots of 1.

PROPOSITION 4: *Fix an integer $n \geq 2$, and z a primitive n-th root of 1. Let π be a representation of Γ on a Hilbert space $\mathcal{H}$.*

(1) *$z \in \mathrm{Sp}\ \pi(h)$ if and only if S is n-coloring and $\chi_{S,z}$ is weakly contained in π.*
(2) *z is an eigenvalue of $\pi(h)$ if and only if S is n-coloring and $\chi_{S,z}$ is contained in π (that is $\mathcal{H}_{S,z}^{=} \neq \{0\}$).*
(3) *If S is n-coloring, if $z \in \mathrm{Sp}\ \pi(h)$ and if $z \notin \mathrm{Sp}\ \pi_{S,z}^{\perp}(h)$, then z is an isolated point of $\mathrm{Sp}\ \pi(h)$.*
(4) *If $\kappa_z(\pi, S) > 0$ then $\mathrm{Sp}\ \pi(h)$ is disjoint from the open ball $\{w \in \mathbb{C}\colon |w-z| < \kappa_z(\pi,S)^2/(2|S|)\}$.*

   Assume moreover that $S = S^{-1}$.
(5) *If S is bicoloring and if -1 is isolated in $\mathrm{Sp}\ \pi(h)$, then $-1 \notin \mathrm{Sp}\ \pi_{S,-1}^{\perp}(h)$.*
(6) *If $\mathrm{Sp}\ \pi(h) \subset [-1+\varepsilon, 1]$ for some $\varepsilon > 0$, then $\kappa_{-1}(\pi, S) \geq \sqrt{2\varepsilon}$.*

Proof: (1) is just a special case of Lemma 6. We leave the details of (2) to (5) to the reader (see Proposition I). For (6), observe that

$$\mathrm{Re}\langle \xi | \pi(s)\xi \rangle) \geq -1 + \varepsilon$$

implies

$$\|\pi(s)\xi + \xi\|^2 = 2(1 + \mathrm{Re}\langle \xi|\pi(s)\xi\rangle) \geq 2\varepsilon. \quad \blacksquare$$

A CONSEQUENCE OF PROPOSITIONS I(1) AND 4(1).

Recall that a group Γ is amenable if and only if its regular representation λ weakly contains some finite dimensional representation [Fel: Theorem 3]. Consequently, for Γ and S as usual, the following are equivalent.

(i) Γ is not amenable.
(ii) $\mathrm{Sp}\ \lambda(h)$ is disjoint from $\{1\}$.
(iii) $\mathrm{Sp}\ \lambda(h)$ is disjoint from $\{1, -1\}$.

There is a straightforward analogue of Proposition 4 involving a complex number $z \in \mathbb{T}$ and a character $\chi_{S,z}\colon \Gamma \to \mathbb{T}$ such that $\chi_{S,z}(S) = \{z\}$. It follows that (i), (ii), (iii) above are equivalent to

(iv) Sp $\lambda(h)$ is disjoint from $\mathbb{T}$.

C. Symmetric spectra

Let $\mathcal{G}$ now be an **oriented finite graph.** One defines naturally an (in general not symmetric) adjacency matrix $A = (A_{v,w})_{v,w\in\mathcal{G}^0}$ for $\mathcal{G}$. Suppose that $\mathcal{G}$ is connnected by oriented paths, namely that the matrix A is irreducible in the sense of Perron–Frobenius theory. It is known that the spectral radius ρ of A is a simple eigenvalue of A, and moreover that the following are equivalent:

(i) $\mathcal{G}$ is bicolorable,

(ii) $-\rho \in \mathrm{Sp}\, A$,

(iii) $\mu \in \mathrm{Sp}\, A \Leftrightarrow -\mu \in \mathrm{Sp}\, A$.

(See e.g. Chapter XIII of [Gan].)

The purpose of this section is to start to investigate how this could carry over to the situation of the previous sections.

Given a self-adjoint operator x on a Hilbert space $\mathcal{H}$, we set

$$\min\ \mathrm{Sp}\ x = \min\{\alpha\colon \alpha \in \mathrm{Sp}\ x\} = \min\{\langle\xi|x\xi\rangle\colon \xi \in \mathbb{S}^1(\mathcal{H})\},$$
$$\max\ \mathrm{Sp}\ x = \max\{\alpha\colon \alpha \in \mathrm{Sp}\ x\} = \max\{\langle\xi|x\xi\rangle\colon \xi \in \mathbb{S}^1(\mathcal{H})\}.$$

Let again Γ, S and h be as in Section A.

PROPOSITION 5: *We assume that Γ is not reduced to one element.*

(1) *If the set S is bicoloring, then the spectrum of $\lambda(h)$ is symmetric with respect to 0.*

(2) *The following are equivalent:*

(i) *S in bicoloring.*

(ii) *The spectrum of h is symmetric with respect to 0.*

(iii) $-1 \in \mathrm{Sp}\ h$.

Suppose moreover that $S^{-1} = S$.

(3) *One has*

$$\max \mathrm{Sp}\ h \geq \max \mathrm{Sp}\ \lambda(h) \geq \frac{1}{|S|} > 0$$

and

$$\min \mathrm{Sp}\ h \leq \min \mathrm{Sp}\ \lambda(h) \leq 0.$$

(4) *If $1 \notin S$, then one has moreover*

$$\min \mathrm{Sp}\ h \leq \min \mathrm{Sp}\ \lambda(h) \leq -\frac{1}{|S|} < 0.$$

(5) *(Due to Kesten.) One has*

$$|\min \mathrm{Sp}\ \lambda(h)| \leq \max \mathrm{Sp}\ \lambda(h).$$

(6) *(Due to Kesten.) If* $S^{-1} = S$ *one has*

$$\max \mathrm{Sp}\ \lambda(h) \geq 2\frac{\sqrt{|S|-1}}{|S|}.$$

If $|S| > 2$ *then one has equality if and only if* Γ *is free on a set* S_+ *of generators such that* $S = S_+ \cup (S_+)^{-1}$.

Proof:

(1) This is a consequence of Lemma 7.

(2) This follows from Lemma 7 and Proposition 3 (2).

As usual, we denote by $(\delta_\gamma)_{\gamma\in\Gamma}$ the canonical basis of $l^2(\Gamma)$. Suppose first that Γ is finite and $\Gamma = S$. Then $\lambda(h)$ is the projection from $l^2(\Gamma)$ onto the line $\mathbf{C}\sum_{\gamma\in\Gamma}\delta_\gamma$, so that $\mathrm{Sp}\ \lambda(h) = \{0,1\}$.

To prove (3) and (4), we may assume from now on that $\Gamma \neq S$. As S generates Γ, and as $S^{-1} = S$, it follows that S is not a subgroup of Γ, so that there exist two (not necessarily distinct) elements s, $t \in S$ such that $st \notin S$. We define the vectors

$$\eta_+ = \frac{1}{\sqrt{2}}(\delta_t + \delta_{st})$$

and

$$\eta_- = \frac{1}{\sqrt{2}}(\delta_t - \delta_{st})$$

in $\mathbb{S}^1(l^2(\Gamma))$. We have

$$\langle\lambda(h)\eta_+|\eta_+\rangle = \frac{1}{2}\{\langle\lambda(h)\delta_t|\delta_t\rangle + \langle\lambda(h)\delta_t|\delta_{st}\rangle + \langle\lambda(h)\delta_{st}|\delta_t\rangle + \langle\lambda(h)\delta_{st}|\delta_{st}\rangle\}.$$

The right hand side is equal to

$$\frac{1}{2}\left\{0 + \frac{1}{|S|} + \frac{1}{|S|} + 0\right\} = 1/|S|,$$

if $1 \notin S$ and to $2/|S|$ if $1 \in S$. Thus

$$\max \mathrm{Sp}\ \lambda(h) \geq 1/|S|$$

in all cases.

Similarly $\langle \lambda(h)\eta_- | \eta_- \rangle$ is equal to $-1/|S|$ if $1 \notin S$ and to 0 if $1 \in S$, so that $\min \mathrm{Sp}\, \lambda(h)$ is bounded above by $-1/|S|$ if $1 \notin S$ and by 0 in all cases. This completes the proof of Claims (3) and (4).

Let us now show (5), which is due to Kesten [Ke1, Formula 2.11]. Let $\{E_t\}_{t\in\mathbb{R}}$ be the spectral resolution of $\lambda(h)$ (see e.g. [RSN: no. 107]), and let σ be the probability Borel measure defined on $\mathbb{R}$ by

$$\sigma(B) = \int_B d\langle E_t\delta_1 | \delta_1 \rangle$$

for all Borel subsets $B \subset \mathbb{R}$, where $\delta_1 \in l^2(\Gamma)$ denotes the Dirac function of support $\{1\} \subset \Gamma$.

Let us first observe that the spectrum $\mathrm{Sp}\, \lambda(h)$ coincides with the support $\mathrm{Supp}\, \sigma$. Indeed, let $\alpha \in \mathbb{R}$. Then

$\alpha \in \mathrm{Sp}\, \lambda(h) \Leftrightarrow$ for all $\epsilon > 0$ one has $E(]\alpha - \epsilon, \alpha + \epsilon[) \neq 0$, by definition of the projection-valued measure E associated to $\{E_t\}_{t\in\mathbb{R}}$

$\Leftrightarrow$ for all $\epsilon > 0$ there exists $\gamma \in \Gamma$ such that $\langle E(]\alpha - \varepsilon, \alpha + \varepsilon[)\delta_\gamma | \delta_\gamma \rangle \neq 0$, because a non-zero projection has some non-zero diagonal matrix element

$\Leftrightarrow$ for all $\varepsilon > 0$ one has $\langle E(]\alpha - \varepsilon, \alpha + \varepsilon[)\delta_1 | \delta_1 \rangle \neq 0$, because $\lambda(h)$ and $E(]\alpha - \varepsilon, \alpha + \varepsilon[)$ commute with the right regular representation of Γ

$\Leftrightarrow \alpha \in \mathrm{Supp}\, \sigma$, by definition of σ.

Let us also observe that all moments of σ are non-negative. Indeed, for each $n \geq 0$ the n-th moment of σ is

$$\mu_n = \int_{\mathbb{R}} t^n d\sigma(t) = \langle \lambda(h)^n \delta_1 | \delta_1 \rangle.$$

As $|S|\lambda(h)$ is the adjacency matrix of the Cayley graph $\mathcal{G}(\Gamma, S)$, the number $\langle |S|^n \lambda(h)^n \delta_1 | \delta_1 \rangle$ is also the number of closed paths of length n in $\mathcal{G}(\Gamma, S)$ starting at 1, so that $\mu_n \geq 0$. (Otherwise said: μ_n is the probability that the appropriate random walk starting at 1 goes back to 1 after n steps, so that $\mu_n \geq 0$.)

Claim (5) is now a consequence of the following standard lemma, from measure theory. For claim (6) we refer to [Ke1]. ∎

LEMMA 8: *Let σ be a probability Borel measure with compact support on the real line. Set*

$$-m = \min\{t \in \mathbb{R}: t \in \operatorname{Supp} \sigma\},$$
$$M = \max\{t \in \mathbb{R}: t \in \operatorname{Supp} \sigma\}.$$

Assume that one has

$$-m < 0 < M,$$
$$\mu_n = \int_{\mathbb{R}} t^n d\sigma(t) \geq 0 \quad \textit{for all } n \in \mathbb{N}.$$

Then one has

$$\limsup_{n\to\infty} (\mu_n)^{1/n} = M = \max(m, M).$$

Proof: Set $F(t) = \int_{-\infty}^{t} d\sigma(t)$ for all $t \in \mathbb{R}$.

We claim first that

$$\limsup_{n\to\infty} (\mu_n)^{1/n} \geq \max(m, M).$$

For each integer $n \geq 0$, one has

$$\mu_n \leq \int_{\mathbb{R}} |t|^n d\sigma(t) \leq \max(m^n, M^n)[F(M+1) - F(m-1)].$$

As the square bracket is strictly positive by definition of m and M, this implies the claim.

Let us now consider the moments of even order. For each $n \geq 0$ and for each small enough real number $\varepsilon > 0$, one has

$$\begin{aligned}\mu_n &\geq \max\left\{\int_{-m-\varepsilon}^{-m+\varepsilon} t^{2n} d\sigma(t), \int_{M-\varepsilon}^{M+\varepsilon} t^{2n} d\sigma(t)\right\} \\ &\geq \max\Big\{(m-\varepsilon)^{2n}[F(-m+\varepsilon) - F(-m-\varepsilon)], \\ &\qquad (M-\varepsilon)^{2n}[F(M+\varepsilon) - F(M-\varepsilon)]\Big\}.\end{aligned}$$

As the square brackets are strictly positive, this implies

$$\limsup_{n\to\infty} (\mu_{2n})^{1/2n} \geq \max\{m-\varepsilon, M-\varepsilon\}.$$

As this holds for all small enough ε, the left-hand side is also larger than $\max\{m, M\}$, so that one has

$$\limsup_{n\to\infty} (\mu_n)^{1/n} = \max(m, M).$$

Let us finally consider the moments of odd order. For each $n \geq 0$ and $\varepsilon > 0$ as above, one has

$$\mu_{2n+1} = \int_{-m-\varepsilon}^{0} t^{2n+1} d\sigma(t) + \int_{0}^{M+\varepsilon} t^{2n+1} d\sigma(t) \geq 0$$

so that

$$(M+\varepsilon)^{2n+1}[F(M+\varepsilon) - F(0)] \geq \int_{0}^{M+\varepsilon} t^{2n+1} d\sigma(t)$$

$$\geq \int_{-m-\varepsilon}^{-m+\varepsilon} |t|^{2n+1} d\sigma(t) \geq (m-\varepsilon)^{2n+1}[F(-m+\varepsilon) - F(-m-\varepsilon)].$$

This implies again $M + \varepsilon \geq m - \varepsilon$, so that one has finally $M \geq m$. ∎

Remarks: (1) Suppose that S is symmetric. If S is bicoloring, then

$$-\min \mathrm{Sp}\, \lambda(h) = \max \mathrm{Sp}\, \lambda(h)$$

by Proposition 5 (1). Conversely, answering a question in a preliminary version of our paper, D.I. Cartwright has shown [Car] that the equality above implies that S is bicoloring.

D. A characterization of discrete Kazhdan groups

We now prove Proposition III from the introduction.

(1) The first assertion follows from Proposition I (4) for $z = 1$, and from Proposition 4 (4) for each other z on the peripheral spectrum of h; indeed, the sum of all (equivalence classes of) irreducible representations of Γ defines a faithful $*$-representation of $C^*(\Gamma)$ [Ped: 4.3.7]. To check the second assertion we notice that, by the first assertion, the distance between two distinct elements in the peripheral spectrum of h is at least ε. Since the angle subtended by an interval of length ε inscribed in $\mathbb{T}$ is $2\mathrm{Arcsin}\frac{\varepsilon}{2}$, we see that the peripheral spectrum has at most $\pi/\mathrm{Arcsin}\frac{\varepsilon}{2}$ points.

(2) If $\mathrm{Sp}\, h \subset [-1, 1-\varepsilon] \cup \{1\}$, the conclusion follows from Proposition I (6); if $-1 \in \mathrm{Sp}\, h \subset \{-1\} \cup [-1+\varepsilon, 1]$, then S is bicoloring by Proposition 3 (2), and the conclusion follows from Proposition 4 (6), and the fact that $\kappa_{-1}(\pi, S) = \kappa(\chi_{S,-1}\pi, S)$.

QUESTIONS:

(1) *Are Kazhdan groups characterized by the fact that 1 is an isolated point of* Sp h *in the case when* S *is not symmetric?*

(2) *Assume that* Γ *has property (T). Consider a symmetric set of generators* $S^{-1} = S$. *Let*

$$\varepsilon_- = 1 + \min\{\alpha \in \mathrm{Sp}\ h\colon \alpha > -1\},$$
$$\varepsilon_+ = 1 - \max\{\alpha \in \mathrm{Sp}\ h\colon \alpha < 1\},$$

so that

$$\{-1+\varepsilon_-, 1-\varepsilon_+\} \subset \mathrm{Sp}\ h \subset \{-1\} \cup [-1+\varepsilon_-, 1-\varepsilon_+] \cup \{1\}.$$

If S *is bicoloring, then Proposition 5 (2) shows that* $\varepsilon_- = \varepsilon_+$. *How do* ε_- *and* ε_+ *compare in general?*

Remarks:

(1) It has been shown in [Val] that a locally compact group G is a Kazhdan group if and only if there is a (necessarily unique) projection p_G in $C^*(G)$ which is annihilated by every irreducible representation of G distinct from χ_1. If Γ is a countable Kazhdan group, S is a symmetric finite generating subset of Γ, and h is as usual , then 1 is isolated in Sp h by Proposition III (1). It also follows from Proposition I (6) that the spectral projection of h corresponding to 1 is the projection p_Γ mentioned above.

(2) If S is symmetric then everything in Proposition III (2) can be reformulated in terms of the Laplacian $\Delta = |S|(1-h)$. For example, if Γ is a Kazhdan group then $\mathrm{Sp}\ \Delta \subset \{0\} \cup [\frac{1}{2}\hat{\kappa}(\Gamma, S)^2, 2|S|]$. Conversely if $\mathrm{Sp}\ \Delta \subset \{0\} \cup [|S|\varepsilon, 2|S|]$ for some $\varepsilon > 0$, then Γ is a Kazhdan group and $\kappa(\Gamma, S) \geq \sqrt{2\varepsilon}$. ∎

Denote by $\mu_1(\mathcal{G})$ the smallest positive eigenvalue of the Laplacian of a finite graph $\mathcal{G}$.

COROLLARY TO PROPOSITION III: *Let* Γ *be a discrete Kazhdan group with a finite set* $S = S^{-1}$ *of generators. Let* $\hat{\kappa} = \hat{\kappa}(\Gamma, S)$. *Let* φ *be a homomorphism of* Γ *onto a finite group* Γ_0 *whose restriction to* $S \cup \{1\}$ *is injective. Consider the Cayley graph* $\mathcal{G} = \mathcal{G}(\Gamma_0, \varphi(S))$. *Then* $\mu_1(\mathcal{G}) \geq \hat{\kappa}^2/2$.

Proof: If λ_0 is the regular representation of Γ_0 on $l^2(\Gamma_0)$, then $\lambda_0 \circ \varphi(\Delta)$ is the combinatorial Laplacian of $\mathcal{G}$. The conclusion follows from the above Remark, since

$$\mathrm{Sp}\ \lambda_0 \circ \varphi(\Delta) \subset \mathrm{Sp}\ \Delta. \quad \blacksquare$$

Remarks:

(1) $\mu_1(\mathcal{G})$ provides qualitative information about the graph. For example, graphs with large μ_1 tend to have large connectivity and small diameter. See [Bie] for a survey of results in this area. The above Corollary is a slight improvement on Lemma 2.3 of [AlM], since the inequality $\hat{\kappa}(\Gamma, S) \geq \kappa(\Gamma, S)$ may be strict, even for finite groups. Also $\hat{\kappa}(\Gamma, S)$ is easier to calculate.

(2) M. Burger has obtained inequalities related to some Kazhdan constants for $\mathrm{SL}(3, \mathbb{Z})$ [Bur]. These can be combined with the Corollary to give a lower bound for μ_1 when

$$\Gamma = \Gamma_0 = \mathrm{SL}(3, \mathbb{Z}/N\mathbb{Z}) \quad (N \geq 2)$$

and S is the set of all matrices of the form

$$\begin{bmatrix} 1 & 2 & j \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & j \\ 0 & 0 & 1 \end{bmatrix}, \quad j = -1, 0, 1$$

and their inverses. The result is that for the corresponding Cayley graph $\mathcal{G}$,

$$\mu_1(\mathcal{G}) \geq (1 - n^{-1/2})^2/32$$

where n is the product of the distinct prime factors of N.

E. Finite dimensional irreducible representations of a Kazhdan group

The aim of this section is to prove Proposition IV from the introduction.

We first need the following simple result.

LEMMA 9: *Let Γ be a finitely generated group with finite generating set S. If π_1 and π_2 are irreducible representations of Γ and 1 is an eigenvalue of the operator $(\pi_1 \otimes \bar{\pi}_2)(h)$, then π_1 and π_2 are equivalent finite dimensional representations.*

Proof: Let $\mathcal{H}_1, \mathcal{H}_2$ be the representation spaces of π_1, π_2 respectively. Using the usual identification of $\mathcal{H}_1 \otimes \bar{\mathcal{H}}_2$ with the space of Hilbert–Schmidt operators from $\mathcal{H}_2$ into $\mathcal{H}_1$, we write $\pi_1 \otimes \bar{\pi}_2$ in the form

$$(\pi_1 \otimes \bar{\pi}_2)(\gamma)\xi = \pi_1(\gamma)\xi\pi_2(\gamma)^{-1}$$

where $\xi\colon \mathcal{H}_2 \to \mathcal{H}_1$ is Hilbert–Schmidt.

Choose ξ with Hilbert–Schmidt norm $\|\xi\|_2 = 1$ such that $(\pi_1 \otimes \bar{\pi}_2)(h)\xi = \xi$. That is

$$\frac{1}{|S|}\sum_{s\in S}(\pi_1 \otimes \bar{\pi}_2)(s)\xi = \xi.$$

By Lemma 3, we have $(\pi_1 \otimes \bar{\pi}_2)(s)\xi = \xi$ for all $s \in S$. It follows that $\pi_1(\gamma)\xi = \xi\pi_2(\gamma)$, for all $\gamma \in \Gamma$. Since π_1, π_2 are irreducible, $\pi_1 \simeq \pi_2$ and ξ is an isomorphism. Since ξ is a compact operator, π_1, π_2 are necessarily finite dimensional. ∎

Remark: A similar argument shows that a (possibly reducible) unitary representation π of Γ has a finite dimensional subrepresentation if and only if 1 is an eigenvalue of $(\pi \otimes \bar{\pi})(h)$. ∎

Now let $\mathcal{H} = \mathbb{C}^m$ be a fixed finite dimensional Hilbert space. Denote by $\|x\|_2 = \mathrm{tr}(x^*x)^{1/2}$ the Hilbert–Schmidt norm of an operator x on $\mathcal{H}$. The next result is corollary 2 of [Was].

LEMMA 10: *Let Γ be a discrete Kazhdan group, let S be a finite set of generators of Γ and set*

$$\varepsilon = \frac{1}{2|S|}\hat{\kappa}(\Gamma, S)^2.$$

If π_1, π_2 are irreducible representations of Γ on $\mathcal{H} = \mathbb{C}^m$ such that $\|\pi_1(s) - \pi_2(s)\|_2 < \varepsilon\sqrt{m}$, for all $s \in S$, then π_1 is equivalent to π_2.

Proof: We may assume that $S = S^{-1}$, since

$$\|\pi_1(s^{-1}) - \pi_2(s^{-1})\|_2 = \|\pi_1(s) - \pi_2(s)\|_2.$$

If $I\colon \mathcal{H} \to \mathcal{H}$ is the identity map, then

$$\begin{aligned}
\|I - (\pi_1 \otimes \bar{\pi}_2)(h)(I)\|_2 &= \left\| I - \frac{1}{|S|}\sum_{s\in S}\pi_1(s)\pi_2(s)^{-1}\right\|_2 \\
&\le \frac{1}{|S|}\sum_{s\in S}\|I - \pi_1(s)\pi_2(s)^{-1}\|_2 \\
&= \frac{1}{|S|}\sum_{s\in S}\|\pi_2(s) - \pi_1(s)\|_2 \\
&< \varepsilon\sqrt{m}.
\end{aligned}$$

Let $\eta = m^{-1/2}I$, so that $\|\eta\|_2 = 1$. Then

$$1 - \langle \eta | (\pi_1 \otimes \bar{\pi}_2)(h)\eta \rangle = \langle \eta | \eta - (\pi_1 \otimes \bar{\pi}_2)(h)\eta \rangle < \varepsilon.$$

Therefore $\mathrm{Sp}\,(\pi_1 \otimes \bar{\pi}_2)(h)$ meets $(1-\varepsilon, 1]$. It follows from Proposition III (1) that 1 is an eigenvalue of $(\pi_1 \otimes \bar{\pi}_2)(h)$ and so $\pi_1 \simeq \pi_2$, by the preceding Lemma. ∎

Proof of Proposition IV *from the Introduction:* We continue with the notation of Lemma 10.

Let $n = |S|$ and $S = \{s_1, \ldots, s_n\}$. Define a norm on $B(\mathcal{H})^n$ by

$$\|(x_1, \ldots, x_n)\| = \max_{1 \le j \le n} \|x_j\|_2.$$

Lemma 10 says that $\pi_1 \simeq \pi_2$ whenever π_1, π_2 are irreducible representations of Γ on $\mathcal{H} = \mathbb{C}^m$ satisfying

$$\|(\pi_1(s_1), \ldots, \pi_1(s_n)) - (\pi_2(s_1), \ldots, \pi_2(s_n))\| < \varepsilon\sqrt{m}.$$

Let k be the number of balls of Hilbert–Schmidt radius $\varepsilon\sqrt{m}/2$ which are required to cover the unitary group $U(m)$. Then $U(m)^n$ is covered by k^n balls of radius $\varepsilon\sqrt{m}/2$, so there are at most k^n inequivalent irreducible representations of Γ on $\mathbb{C}^m$. It remains to estimate k .

Now $U(m) \subset \{x \in B(\mathbb{C}^m)\colon \|x\|_2 = \sqrt{m}\}$. Using matrix entries as coordinates, and identifying $\mathbb{C}$ with $\mathbb{R}^2$, we have, for the usual Euclidean norm,

$$U(m) \subset \{x \in \mathbb{R}^{2m^2}\colon \|x\| = \sqrt{m}\}.$$

It is therefore enough to find the number of balls of radius $\varepsilon\sqrt{m}/2$ which are required to cover a sphere of radius $\sqrt{m}$ in $\mathbb{R}^{2m^2}$. This is the same as covering a sphere of radius 1 with balls of radius $\varepsilon/2$. A sharp asymptotic bound for this is given by the Corollary in [Wyn]. There it is shown that $k = e^{\alpha m^2}$ balls are enough to do the job, where $\alpha > 0$ is constant. In fact, for sufficiently large m, we need only choose $\alpha > -2\log\sin\theta$, where θ is the half-angle subtended at 0 by a ball of radius $\varepsilon/2$ with centre on the sphere of radius one. Letting $A = A(\Gamma, S)$ denote the constant αn of the preceding argument, we obtain

$$\mathrm{Irrep}_\Gamma(m) = 0(e^{Am^2}).$$

Remark: There exists a discrete Kazhdan group with no nontrivial finite dimensional representations. This follows from [Gro: Chapter 5]. The result is that any lattice in $\mathrm{Sp}(1,q)$, $q \geq 2$, has uncountably many infinite quotients which are simple and torsion. Such a quotient provides the desired example. For if there is a non-trivial finite dimensional representation then it is faithful (by simplicity), so the group is linear. However a linear non-amenable group has to contain a copy of the free group on two generators by Tits' theorem [Tit]. This contradicts the fact that the group is a torsion group.

Let Γ be the group $\mathrm{SL}(n,\mathbb{Z})$, with $n \geq 3$. Steinberg [Ste] has shown that any finite dimensional unitary representation of Γ factorizes through $\mathrm{SL}(n,\mathbb{Z}/k\mathbb{Z})$ for some integer k. Using this and information on the characters of $\mathrm{SL}(n,\mathbb{Z}/p\mathbb{Z})$ for prime p's, it should be possible to improve the bound of Proposition IV for $\mathrm{SL}(n,\mathbb{Z})$, and also to obtain lower asymptotic bounds for $\mathrm{Irrep}_\Gamma(m)$. ∎

In the line of our work, there are numerous questions which remain open. Let us mention here the following ones:

When is $\mathrm{Sp}(\lambda(h))$ connected?

When does $\lambda(h)$ have eigenvalues besides 1 and -1 ?

When is the spectral measure of $\lambda(h)$ absolutely continuous with respect to the Lebesgue measure on [-1,1] ?

Finally we remark that Proposition I of the present paper has been used in [HRV] to obtain results on exactness for group C^*-algebras.

Note added in proof: In a remarkable piece of work, D. I. Cartwright, W. Mlotkowski and T. Steger, *Property (T) and $\tilde{A}_2$ groups*, preprint, University of Sydney, 1992, the authors show by combinatorial methods that certain groups associated with buildings have property (T). Moreover they calculate the spectrum of h and obtain the exact values of the Kazhdan constants, using the estimates of Proposition I (6).

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